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Noether-Symmetry Analysis using Alternative Lagrangian Representations

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We have sought to work with an approach to Noether symmetry analysis which uses the properties of infinitesimal point transformations in the space-time (q, t) variable to establish the association between symmetries and conservation laws of a dynamical system. In this approach symmetries are expressed in the form of generators. We have studied the variational or Noether symmetries of two uncoupled Harmonic oscillators and two such oscillators coupled by an interaction. Both these systems can have alternative Lagrangian representations. We have studied in detail how the association between symmetries and conservation laws changes as one alters the analytic or Lagrangian representation. This analysis is carried out with a view to explicitly demonstrate that the correlation between symmetry transformation and corresponding invariant quantity depends crucially on the choice of the analytic representation.

KEY WORDS: Newtonian system; alternative Lagrangian representations; Noether symmetry; conservation laws.

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1. INTRODUCTION

In the simplest problem of variational calculus one attempts to find a path $E = E_0$ in the velocity-phase space along which the action functional

$$W(E) = \int_{t_1}^{t_2} L\,(q,\, \dot{q},\, t)\, dt \tag{1}$$

is an extremum. By the term velocity-phase space we mean the space $(q, \dot{q})$ where q is a generalized coordinate and $\dot{q} = \frac{dq}{dt}$, the generalized velocity. In the newtonian context, the functions $L\,(q, \dot{q}, t)$ are of at least class $\mathcal{E}^4$ in a region R^{2n+1} and are often called the admissible Lagrangians (Santilli, 1978). We note

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that the real-valued functions F_i are of class $\mathcal{E}^m$ in a region R^{3n+1} when they posses continuous partial derivatives (with respect to the arguments) upto and including the order m everywhere in R^{3n+1}.

Two admissible Lagrangians are called equivalent iff they are proportional to and/or differ by a guage function. Otherwise, they are called alternative or inequivalent. Evolution of many natural processes admits alternative Lagrangian representations. Interestingly, description of physical systems using alternative Lagrangians can have deep consequences on the further devlopement of the theory. For example, one can come across ambiguities in the association of symmetries with constants of the motion (Morandi *et al.*, 1990). It is well known that such association is provided by Noether's theorem. The object of the present work is to envisage an in-depth study for the relation between symmetries and conservation laws of newtonian systems which can be analytically represented by alternative Lagrangians. We shall express the symmetries in the form of generators and construct the concommitant constants of the motion. We shall also try to interpret our results in physical terms.

In the classical Noether theorem, if a given system of differential equations follows from the variational principle, then a continuous symmetry transformation (point, contact or higher order) that leaves the action functional invariant to within a divergence yields a conservation law. The proof of this theorem requires some knowledge of differential form, Lie derivatives and pull-back (Olver, 1993). Use of similar sophisticated mathematical tools is also required to study the ambiguities in the association of symmetries with constants of the motion. In particular, one needs to work with the geometry of the tangent bundle over a differential manifold (Morandi *et al.*, 1990). In our work we shall, however, carry out the symmetry analysis by using a relatively simpler mathematical framework as compared to that of the algebro-geometric theories (Morandi *et al.*, 1990; Olver, 1993). In fact, we shall make use of some point transformations that depend on time and spatial coordinates. The approach to be followed by us has an old root in the classical-mechanics literature. For example, as early as 1951, Hill (1951) provided a simplified account of Noether's theory by considering infinitesimal transformations of the dependent and independent variables of the particle dynamics or field theory. In the recent past Struckmeier and Riedel (2002) used a similar approach to study the Noether and Lie symmetries for the time-dependent Kepler problem. An obvious virtue of their approach is its simplicity and directness. We shall study the relation between symmetries and conservation laws with special emphasis on (i) two uncoupled Harmonic oscillators and (ii) Harmonic oscillators coupled by an interaction. The equations of motion of these systems can have alternative Lagrangian representations. We are primarily interested to examine how does the association between symmetries and conservation laws change as we go from one Lagrangian representation to the other.

In Section 2 we present the results for the Lagrangians and corresponding Hamiltonians. We outline in Section 3 our scheme for symmetry analysis by the use of Noether's theorem. We devote Sections 4 and 5 to present the main results of this work for the relation between symmetries and conservation laws in the presence of alternative Lagrangians. Our results also include the generators of the symmetry transformations together with the algebra satisfied by them. Moreover, we present all appropriate results for constants of the motion. Finally, in Section 6, we summarize our outlook on the present work.

2. ANALYTIC REPRESENTATIONS IN CONFIGURATION – AND PHASE SPACE

The representation of physical systems in terms of Lagrangians and Hamiltonians often goes by the name analytic representation (Santilli, 1978). Here we shall construct analytic representations for the systems in (i) and (ii) with a view to use them for symmetry analysis. First we consider the uncoupled oscillators represented by

$$\ddot{q}_1 + \omega^2 q_1 = 0 \tag{2a}$$

and

$$\ddot{q}_2 + \omega^2 q_2 = 0 \tag{2b}$$

with ω, the eigenfrequency of the identical system in (2). Here q_i stand for the generalized coordinates and overdots denote differentiation with respect to time. It is easy to see that (2) can be analytically represented in the configuration space by using the alternative Lagrangians

$$L^d = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \frac{1}{2}\omega^2\left(q_1^2 + q_2^2\right) \tag{3a}$$

and

$$L^a = \dot{q}_1\dot{q}_2 - \omega^2 q_1 q_2. \tag{3b}$$

We have used the superscripts d and a on L to denote direct and alternative representations presumably because L^d when substituted in the Euler-Lagrange equation in q_1 or q_2 gives the equation of motion for q_1 or q_2 while L^a does the opposite. The Euler-Lagrange equation in q_1 gives the equation of motion for q_2 and the conversly. The Lagrangians L^d and L^a are not connected by a gauge term such that the results in (3) refer to the alternative Lagrangians of the uncoupled oscillators in (2). The Hamiltonians for the Lagrangians in (3a) and (3b) are given by

$$H^d = \frac{1}{2}\left({p_1^d}^2 + {p_2^d}^2\right) + \frac{1}{2}\omega^2\left(q_1^2 + q_2^2\right) \tag{4a}$$

and

$$H^a = p_1^a p_2^a + \omega^2 q_1 q_2. \tag{4b}$$

with the canonical momenta $p_1^d = \dot{q}_1$, $p_2^d = \dot{q}_2$, $p_1^a = \dot{q}_2$ and $p_2^a = \dot{q}_1$.

The second system of our interest consists of two one dimensional harmonic oscillators coupled by an interaction $-\alpha q_1 q_2$, with α, the coupling constant. The equations of motion are (Landau and Lifshitz, 1982)

$$\ddot{q}_1 + \omega^2 q_1 = \alpha q_2 \tag{5a}$$

and

$$\ddot{q}_2 + \omega^2 q_2 = \alpha q_1. \tag{5b}$$

As with the equations in (3) the alternative Lagrangians representing (5) can be written as

$$L^d = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \frac{1}{2}\omega^2\left(q_1^2 + q_2^2\right) + \alpha q_1 q_2 \tag{6a}$$

and

$$L^a = \dot{q}_1 \dot{q}_2 - \omega^2 q_1 q_2 + \frac{\alpha}{2}\left(q_1^2 + q_2^2\right). \tag{6b}$$

The corresponding Hamiltonians are given by

$$H^d = \frac{1}{2}\left(p_1^{d^2} + p_2^{d^2}\right) + \frac{1}{2}\omega^2\left(q_1^2 + q_2^2\right) - \alpha q_1 q_2 \tag{7a}$$

and

$$H^a = p_1^a p_2^a + \omega^2 q_1 q_2 - \frac{\alpha}{2}\left(q_1^2 + q_2^2\right). \tag{7b}$$

3. SYMMETRIES AND CONSERVATION LAWS

We shall study the infinitesimal criterion for the invariance of a variational problem under a group of transformations that map ‘points’ in configuration space $(\vec{q}, t)$ into their infinitesimal neighbourhood $(\vec{q}\,', t')$. Here $\vec{q} = \{q_i\},\ i = 1, \ldots, n$, stands for the set of generalized coordinates representing the dynamical system under consideration and, as usual, t is the time parameter. Formally, such point transformations are represented as

$$t' = t + \delta t, \qquad \delta t = \epsilon \xi(\vec{q}, t), \tag{8a}$$

$$q_i{}' = q_i + \delta q_i, \qquad \delta q_i = \epsilon \eta_i(\vec{q}, t) \tag{8b}$$

with ϵ, an infinitesimal parameter. Given the transformation rule for q_i, the corresponding results for $\dot{q}_i$ and $\ddot{q}_i$ are given by Struckmeier and Riedel (2002)

$$\delta \dot{q}_i = \epsilon \left[\dot{\eta}_i(\vec{q}, t) - \dot{\xi}(\vec{q}, t)\dot{q}_i \right] \tag{9}$$

and

$$\delta \ddot{q}_i = \epsilon \left[\ddot{\eta}_i(\vec{q}, t) - 2\dot{\xi}(\vec{q}, t)\ddot{q}_i - \ddot{\xi}(\vec{q}, t)\dot{q}_i \right] . \tag{10}$$

Considering the variation of an arbitrary analytic function $u(\vec{q}, t)$ it is easy to prove that

$$\delta u = \epsilon U u(\vec{q}, t) \tag{11}$$

with

$$U = \xi(\vec{q}, t)\frac{\partial}{\partial t} + \sum_{i=1}^{n} \eta_i(\vec{q}, t)\frac{\partial}{\partial q_i} . \tag{12}$$

The operator U is the generator of the infinitesimal point transformations in (8) and represents a vector field on $(\vec{q}, t)$ since it assigns a tangent vector to each points within $(\vec{q}, t)$. A similar cosideration applied to $v(\vec{q}, \dot{\vec{q}}, t)$ gives

$$\delta v = \epsilon U^{(1)} v(\vec{q}, \dot{\vec{q}}, t) \tag{13}$$

with

$$U^{(1)} = U + \sum_{i=1}^{n} \left(\dot{\eta}_i(\vec{q}, t) - \dot{\xi}(\vec{q}, t)\dot{q}_i \right) \frac{\partial}{\partial \dot{q}_i} . \tag{14}$$

The presence of $\frac{\partial}{\partial \dot{q}_i}$ in (14) clearly shows that $U^{(1)}$ is the first prolongation (Olver, 1993) of U.

To write the Noether's theorem we consider, among the general set of point transformations defined by (8), only those that leave the action Ldt invariant. In other words, we demand that

$$L(\vec{q}_i, \dot{\vec{q}}_i, t) \stackrel{!}{=} L'(\vec{q}_i', \dot{\vec{q}}_i', t'). \tag{15}$$

In order to satisfy the condition in (15), we allow the Lagrangian to change its functional form ($L \to L'$). The functional relation between L' and L may be expressed by introducing a gauge function $f(\vec{q}, t)$ (Hill, 1951; Struckmeier and Riedel, 2002) such that

$$L'(\vec{q}_i', \dot{\vec{q}}_i{}', t') = L(\vec{q}_i', \dot{\vec{q}}_i{}', t') - \epsilon \frac{df(\vec{q}, t)}{dt}. \tag{16}$$

From (15) and (16) we have

$$L(\vec{q}_i', \dot{\vec{q}}_i{}', t')\, dt' = L(\vec{q}_i, \dot{\vec{q}}_i, t)\, dt + \epsilon \frac{df(\vec{q}, t)}{dt} dt. \tag{17}$$

On the other hand using L for v in (13) we have

$$L(\vec{q}_i', \dot{\vec{q}}_i', t') = L(\vec{q}_i, \dot{\vec{q}}_i, t) + \epsilon U^{(1)} L(\vec{q}_i, \dot{\vec{q}}_i, t). \tag{18}$$

From (17) and (18) it is easy to see that

$$\frac{df(\vec{q},t)}{dt} = \dot{\xi} L + \xi \frac{\partial L}{\partial t} + \sum_{i=1}^{n} \left(\eta_i \frac{\partial L}{\partial q_i} + \left(\dot{\eta}_i - \dot{\xi}\dot{q}_i\right) \frac{\partial L}{\partial \dot{q}_i} \right). \tag{19}$$

In writing (19) we have made use of the results in (12) and (14). We, therefore, infer that the action is invariant under those point transformations whose constituents ξ and η_i satisfy (19). The terms in (19) can be rearranged to write

$$\frac{dI}{dt} + \sum_{i=1}^{n} (\xi \dot{q}_i - \eta_i) \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) = 0 \tag{20}$$

with

$$I = \sum_{i=1}^{n} (\xi \dot{q}_i - \eta_i) \frac{\partial L}{\partial \dot{q}_i} - \xi L + f(\vec{q}, t). \tag{21}$$

Along the trajectory of the system, the Euler-Lagrange equations hold good such that the second term in (20) is zero. Thus I given in (21) is a conserved quantity or a constant of the motion. The invariant given in (21) and the differential equations for the gauge function in (19) is commonly known as the Noether theorem.

In the Hamiltonian formulation of classical mechanics the noether's invariant can be written as

$$I = \xi(\vec{q}, t) H(\vec{q}, \vec{p}, t) - \sum_{i=1}^{n} \eta_i(\vec{q}, t) p_i + f(\vec{q}, t). \tag{22}$$

We have obtained (22) form (21) using the relation between H and L as given by the usual Legendre transformation

$$L(\vec{q}, \dot{\vec{q}}, t) = \sum_{i=1}^{n} p_i \dot{q}_i - H(\vec{q}, \vec{p}, t), \quad p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{23}$$

In terms of the Hamiltonian the differential equation (19) now reads

$$\frac{d}{dt} \left[\xi(\vec{q}, t) H(\vec{q}, \vec{p}, t) - \sum_{i=1}^{n} \eta_i(\vec{q}, t) p_i + f(\vec{q}, t) \right] = 0. \tag{24}$$

Clearly, the expression inside the squared bracket in (24) stands for the conserved quantity given in (22). Equation (24) provides a natural basis to carry out Noether symmetry analysis for Newtonian systems.

4. NOETHER SYMMETRIES OF UNCOUPLED HARMONIC OSCILLATORS

4.1. Direct Representation

We first perform the Noether symmetry analysis for the uncoupled Harmonic oscillators represented by (2a) and (2b). Two alternative Lagrangians for the system are given in (3a) and (3b) with the corresponding Hamiltonians being represented by (4a) and (4b) respectively. The superscripts used in these equations are not essential to carry out the analysis. Thus we shall omit them henceforth. For the direct Hamiltonian in (4a), (24) can be written in the form

$$\begin{aligned}\frac{\partial f}{\partial t} &+ p_1\frac{\partial f}{\partial q_1} + p_2\frac{\partial f}{\partial q_2}\\ &+ \frac{1}{2}\left(\frac{\partial \xi}{\partial t} + p_1\frac{\partial \xi}{\partial q_1} + p_2\frac{\partial \xi}{\partial q_2}\right)\left(p_1^2 + p_2^2 + \omega^2 q_1^2 + \omega^2 q_2^2\right)\\ &- \left(\frac{\partial \eta_1}{\partial t} + p_1\frac{\partial \eta_1}{\partial q_1} + p_2\frac{\partial \eta_1}{\partial q_2}\right)p_1 + \omega^2\eta_1 q_1\\ &- \left(\frac{\partial \eta_2}{\partial t} + p_1\frac{\partial \eta_2}{\partial q_1} + p_2\frac{\partial \eta_2}{\partial q_2}\right)p_2 + \omega^2\eta_2 q_2 = 0. \qquad (25)\end{aligned}$$

In writing (25) we have made use of the canonical equations

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = p_1, \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1} = -\omega^2 q_1 \qquad (26a)$$

and

$$\dot{q}_2 = \frac{\partial H}{\partial p_2} = p_2, \quad \dot{p}_2 = -\frac{\partial H}{\partial q_2} = -\omega^2 q_2. \qquad (26b)$$

Equation (25) can be globally satisfied for any particular choice of the momenta provided the sum of momentum-independent terms, the coefficients of linear, quadratic and cubic terms in p_1 and p_2 vanish separately. Following this viewpoint we write

$$p_{1,2}^0: \quad \frac{\partial f}{\partial t} + \frac{\omega^2}{2}\left(q_1^2 + q_2^2\right)\frac{\partial \xi}{\partial t} + \omega^2\eta_1 q_1 + \omega^2\eta_2 q_2 = 0, \qquad (27a)$$

$$p_1: \quad \frac{\partial f}{\partial q_1} + \frac{\omega^2}{2}\left(q_1^2 + q_2^2\right)\frac{\partial \xi}{\partial q_1} - \frac{\partial \eta_1}{\partial t} = 0, \qquad (27b)$$

$$p_2: \quad \frac{\partial f}{\partial q_2} + \frac{\omega^2}{2}\left(q_1^2 + q_2^2\right)\frac{\partial \xi}{\partial q_2} - \frac{\partial \eta_2}{\partial t} = 0, \qquad (27c)$$

$$p_1^2: \quad \frac{1}{2}\frac{\partial \xi}{\partial t} - \frac{\partial \eta_1}{\partial q_1} = 0, \tag{27d}$$

$$p_2^2: \quad \frac{1}{2}\frac{\partial \xi}{\partial t} - \frac{\partial \eta_2}{\partial q_2} = 0, \tag{27e}$$

$$p_1 p_2: \quad -\frac{\partial \eta_1}{\partial q_2} - \frac{\partial \eta_2}{\partial q_1} = 0, \tag{27f}$$

$$p_1 p_2^2: \quad \frac{1}{2}\frac{\partial \xi}{\partial q_1} = 0, \tag{27g}$$

$$p_2 p_1^2: \quad \frac{1}{2}\frac{\partial \xi}{\partial q_2} = 0, \tag{27h}$$

$$p_1^3: \quad \frac{1}{2}\frac{\partial \xi}{\partial q_1} = 0 \tag{27i}$$

and

$$p_2^3: \quad \frac{1}{2}\frac{\partial \xi}{\partial q_2} = 0. \tag{27j}$$

Equation (27a) signifies that we have equated the sum of p-independent terms to zero while (27b)–(27j) have been obtained by equating the sum of the coefficients of p_1, p_2, p_1^2 etc to zero. From (27g)–(27j) we see that ξ is not function of q_1 and q_2 . Thus

$$\xi(q_1, q_2, t) \equiv \xi(t) = \beta(t) \text{ (say)}. \tag{28}$$

Sum of (27d), (27e) and (27f) can be written in a compact form

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\left(\frac{1}{2}\delta_{ij}\frac{\partial \xi}{\partial t} - \frac{\partial \eta_i}{\partial q_j}\right) = 0. \tag{29}$$

Equation (29) will be satisfied globally if $\frac{\partial \eta_i}{\partial q_j}$ cancels the term $\frac{1}{2}\delta_{ij}\dot{\xi}$ up to a constant element a_{ij} of an antisymmetric matrix (a_{ij}) such that

$$\frac{\partial \eta_i}{\partial q_j} = \frac{1}{2}\delta_{ij}\dot{\beta}(t) + a_{ij} \quad a_{ij} = -a_{ji}. \tag{30}$$

In writing (30) we have made use of (28). Equation (30) can be integrated to get

$$\eta_i(\vec{q}, t) = \frac{1}{2}\dot{\beta} q_i + \psi_i(t) + \sum_{j=1}^{2} a_{ij} q_j, \tag{31}$$

where $\psi_i(t)$ is a constant of integration. In view of (28), we can write (27a), (27b) and (27c) as

$$\frac{\partial f}{\partial t} + \frac{\omega^2}{2}\left(q_1^2 + q_2^2\right)\dot{\beta} + \omega^2\eta_1 q_1 + \omega^2\eta_2 q_2 = 0, \tag{32}$$

$$\frac{\partial f}{\partial q_1} - \frac{\partial \eta_1}{\partial t} = 0 \tag{33}$$

and

$$\frac{\partial f}{\partial q_2} - \frac{\partial \eta_2}{\partial t} = 0. \tag{34}$$

For $\eta_i(\vec{q}, t)$ in (31), we see that

$$f = \frac{1}{4}q_1^2\ddot{\beta} + \frac{1}{4}q_2^2\ddot{\beta} + \dot{\psi}_1 q_1 + \dot{\psi}_2 q_2 \tag{35}$$

represents a general solution of (33) and (34). Using the expressions for η_i and f from (31) and (35) in (22) we obtain the invariant I in the form

$$I = I_\beta + I_{\psi_1} + I_{\psi_2} + I_a, \tag{36}$$

where

$$I_\beta = \frac{1}{4}\left(q_1^2 + q_2^2\right)\ddot{\beta} - \frac{1}{2}\left(q_1 p_1 + q_2 p_2\right)\dot{\beta}$$

$$+\frac{1}{2}\left(p_1^2 + p_2^2 + \omega^2 q_1^2 + \omega^2 q_2^2\right)\beta, \tag{37a}$$

$$I_{\psi_i} = \dot{\psi}_i q_i - \psi_i p_i, \qquad i = 1,\ 2 \tag{37b}$$

and

$$I_a = -a_{12} q_2 p_1 - a_{21} q_1 p_2. \tag{37c}$$

In writing (36) we also used (4a) and (28). Each of the I's in (37) is expected to form a separate constant. This can be seen as follows.

Substituting the values of η_i and f in (32) we get

$$J_\beta + J_{\psi_1} + J_{\psi_2} + J_a = 0, \tag{38}$$

where

$$J_\beta = \left(q_1^2 + q_2^2\right)\left(\frac{1}{4}\dddot{\beta} + \omega^2\dot{\beta}\right), \tag{39a}$$

$$J_{\psi_i} = \left(\ddot{\psi}_i + \omega^2\psi_i\right)q_i, \ \ i = 1,\ 2 \tag{39b}$$

and

$$J_a = \omega^2 \left(a_{12} q_1 q_2 + a_{21} q_1 q_2 \right). \tag{39c}$$

Using the appropriate Hamilton's equations it is easy to verify that

$$\int J_\beta dt = I_\beta \tag{40a}$$

and

$$\int J_{\psi_i} dt = I_{\psi_i}, \ i = 1, 2. \tag{40b}$$

Equations (40a) and (40b) verify our conjecture. The matrix $\left(a_{ij}\right)$ is antisymmetric. Therefore $a_{11} = a_{22} = 0$ and $a_{12} = -a_{21}$. Thus for the two dimensional case under consideration $\left(a_{ij}\right)$ can not contain more than one independent element. In view of this (39c) becomes identically equal to zero and (37c) gives

$$I_a = q_1 p_2 - q_2 p_1 \qquad \text{for} \quad a_{12} = 1. \tag{40c}$$

The special values of $\beta(t)$ and $\psi_i(t)$ can be obtained from

$$J_\beta = 0 \tag{41a}$$

and

$$J_{\psi_i} = 0. \tag{41b}$$

Equations (41a) and (41b) give

$$\beta = 1 \ \text{ and } \ \beta^{\pm} = e^{\pm 2 i \omega t} \tag{42a}$$

and

$$\psi_i^{\pm} = e^{\pm i \omega t}, \ i = 1, 2. \tag{42b}$$

The generators of the symmetry transformations leading to the conserved quantities in (37) can be obtained by using the values of $\xi(t)$ and $\eta_i(\vec{q}\,, t)$ from (28) and (31) into (12). Thus we have

$$U = U_\beta + U_{\psi_i} + U_a, \tag{43}$$

where

$$U_\beta = \beta \frac{\partial}{\partial t} + \frac{1}{2} q_1 \dot{\beta} \frac{\partial}{\partial q_1} + \frac{1}{2} q_2 \dot{\beta} \frac{\partial}{\partial q_2}, \tag{44a}$$

$$U_{\psi_i} = \psi_i \frac{\partial}{\partial q_i}, \ \ i = 1, 2 \tag{44b}$$

and

$$U_a = q_2 \frac{\partial}{\partial q_1} - q_1 \frac{\partial}{\partial q_2}. \tag{44c}$$

For $\beta = 1$, the invariant (37a) reduces to

$$I_{\beta=1} = H(= H_S). \tag{45a}$$

Understandably, (45a) represents the well known result that the instantaneous system energy is given by H when the Hamiltonian does not depend on time explicitly. The corresponding generator of the symmetry transformation from (44a) is the time translation operator

$$U_{\beta=1} = \frac{\partial}{\partial t}. \tag{46a}$$

For $\beta = e^{+2i\omega t}$, the invariant I_β gives rise to two real invariants

$$I_{\beta^1} = \mathrm{Re} I_{\beta=e^{+2i\omega t}} = \frac{1}{2}\left(p_1^2 + p_2^2 - \omega^2 q_1^2 - \omega^2 q_2^2\right) \cos 2\omega t$$
$$+\omega\left(q_1 p_1 + q_2 p_2\right) \sin 2\omega t \tag{45b}$$

and

$$I_{\beta^2} = \mathrm{Im} I_{\beta=e^{+2i\omega t}} = \frac{1}{2}\left(p_1^2 + p_2^2 - \omega^2 q_1^2 - \omega^2 q_2^2\right) \sin 2\omega t$$
$$-\omega\left(q_1 p_1 + q_2 p_2\right) \cos 2\omega t. \tag{45c}$$

The generators of I_{β^1} and I_{β^2} as found from (44a) are given by

$$\begin{aligned} U_{\beta^1} &= \mathrm{Re} U_{\beta=e^{+2i\omega t}} \\ &= \cos 2\omega t \frac{\partial}{\partial t} - \omega q_1 \sin 2\omega t \frac{\partial}{\partial q_1} - \omega q_2 \sin 2\omega t \frac{\partial}{\partial q_2} \end{aligned} \tag{46b}$$

and

$$\begin{aligned} U_{\beta^2} &= \mathrm{Im} U_{\beta=e^{+2i\omega t}} \\ &= \sin 2\omega t \frac{\partial}{\partial t} + \omega q_1 \cos 2\omega t \frac{\partial}{\partial q_1} + \omega q_2 \cos 2\omega t \frac{\partial}{\partial q_2}. \end{aligned} \tag{46c}$$

For $\beta = e^{-2i\omega t}$, the results similar to those in (45b), (45c) and (46b), (46c) read

$$I_{\beta^3} = \mathrm{Re} I_{\beta=e^{-2i\omega t}} = I_{\beta^1}, \tag{45d}$$

$$I_{\beta^4} = \mathrm{Im} I_{\beta=e^{-2i\omega t}} = -I_{\beta^2} \tag{45e}$$

and

$$U_{\beta^3} = \mathrm{Re}U_{\beta=e^{-2i\omega t}} = U_{\beta^1}, \tag{46d}$$

$$U_{\beta^4} = \mathrm{Im}U_{\beta=e^{-2i\omega t}} = -U_{\beta^2}. \tag{46e}$$

From (37b), (42b) and (44b) we get the following invariants and generators.

$$I_{\Psi_1^1} = \mathrm{Re}I_{\psi_1=e^{+i\omega t}} = -p_1\cos\omega t - \omega q_1 \sin\omega t, \tag{45f}$$

$$I_{\Psi_1^2} = \mathrm{Im}I_{\psi_1=e^{+i\omega t}} = -p_1\sin\omega t + \omega q_1 \cos\omega t, \tag{45g}$$

$$U_{\Psi_1^1} = \mathrm{Re}U_{\psi_1=e^{+i\omega t}} = \cos\omega t \frac{\partial}{\partial q_1}, \tag{46f}$$

$$U_{\Psi_1^2} = \mathrm{Im}U_{\psi_1=e^{+i\omega t}} = \sin\omega t \frac{\partial}{\partial q_1}, \tag{46g}$$

$$I_{\Psi_1^3} = \mathrm{Re}I_{\psi_1=e^{-i\omega t}} = I_{\Psi_1^1}, \tag{45h}$$

$$I_{\Psi_1^4} = \mathrm{Im}I_{\psi_1=e^{-i\omega t}} = -I_{\Psi_1^2}, \tag{45i}$$

$$U_{\Psi_1^3} = \mathrm{Re}U_{\psi_1=e^{-i\omega t}} = U_{\Psi_1^1}, \tag{46h}$$

$$U_{\Psi_1^4} = \mathrm{Im}U_{\psi_1=e^{-i\omega t}} = -U_{\Psi_1^2}, \tag{46i}$$

$$I_{\Psi_2^1} = \mathrm{Re}I_{\psi_2=e^{+i\omega t}} = -p_2\cos\omega t - \omega q_2 \sin\omega t, \tag{45j}$$

$$I_{\Psi_2^2} = \mathrm{Im}I_{\psi_2=e^{+i\omega t}} = -p_2\sin\omega t + \omega q_2 \cos\omega t, \tag{45k}$$

$$U_{\Psi_2^1} = \mathrm{Re}U_{\psi_2=e^{+i\omega t}} = \cos\omega t \frac{\partial}{\partial q_2}, \tag{46j}$$

$$U_{\Psi_2^2} = \mathrm{Im}U_{\psi_2=e^{+i\omega t}} = \sin\omega t \frac{\partial}{\partial q_2}, \tag{46k}$$

$$I_{\Psi_2^3} = \mathrm{Re}I_{\psi_2=e^{-i\omega t}} = I_{\Psi_2^1}, \tag{45l}$$

$$I_{\Psi_2^4} = \mathrm{Im}I_{\psi_2=e^{-i\omega t}} = -I_{\Psi_2^2}, \tag{45m}$$

$$U_{\Psi_2^3} = \mathrm{Re}U_{\psi_2=e^{-i\omega t}} = U_{\Psi_2^1} \tag{46l}$$

and

$$U_{\Psi_2^4} = \mathrm{Im}U_{\psi_2=e^{-i\omega t}} = -U_{\Psi_2^2}. \tag{46m}$$

Table I. Commutation Relations for the Generators in (47). Each element G_{ij} in the Table Being Represented by $G_{ij} = [G_i\ ,\ G_j]$

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
G_1	0	$2\omega G_7$	ωG_4	ωG_3	ωG_6	ωG_5	$2\omega G_2$	0
G_2	$-2\omega G_7$	0	$-\omega G_3$	ωG_4	$-\omega G_5$	ωG_6	$-2\omega G_1$	0
G_3	$-\omega G_4$	ωG_3	0	0	0	0	ωG_4	$-G_5$
G_4	$-\omega G_3$	$-\omega G_4$	0	0	0	0	$-\omega G_3$	$-G_6$
G_5	$-\omega G_6$	ωG_5	0	0	0	0	ωG_6	G_3
G_6	$-\omega G_5$	$-\omega G_6$	0	0	0	0	$-\omega G_5$	G_4
G_7	$-2\omega G_2$	$2\omega G_1$	$-\omega G_4$	ωG_3	$-\omega G_6$	ωG_5	0	0
G_8	0	0	G_5	G_6	$-G_3$	$-G_4$	0	0

In the above the odd and even superscripts on β and ψ_i refer to real and imaginary part of the invariants and the generators as the case may be. From (44c) and (46a)–(46m) we find that there are only eight linearly independent group generators given by

$$G_1 = U_{\beta^1},\ \ G_2 = U_{\beta^2},\ \ G_3 = U_{\Psi_1^1} \text{ and } G_4 = U_{\Psi_1^2}. \tag{47a}$$

$$G_5 = U_{\Psi_2^1},\ \ G_6 = U_{\Psi_2^2},\ \ G_7 = U_{\beta=1} \text{and } G_8 = U_a. \tag{47b}$$

We have already point out that G_7 represents the generator of the symmetry transformation that conserves the total energy of the system. We further note that G_8 is a generator that arises due to rotation in the $(q_1\ ,\ q_2)$ plane. The system is rotationally invariant and the corresponding conserved quantity is the angular momentum given in (40c). For $q_2 = q_1$, equations in (2a) and (2b) reduce to the equation for a single oscillator. In this case, the generators G_5 and G_6 coalesce with G_3 and G_4 respectively. The generator G_8 vanishes altogether. This leaves us with only five linearly independent group generators of the one dimensional Harmonic oscillator (Lutzky, 1978). The algebra of our eight parameter Lie group is given in Table I.

To each of the one parameter subgroups in Table I there corresponds a constant of the motion (C_i). More explicitly, we write

$$C_1 = I_{\beta^1},\ \ C_2 = I_{\beta^2},\ \ C_3 = I_{\Psi_1^1} \ \text{ and } C_4 = I_{\Psi_1^2}. \tag{48a}$$

$$C_5 = I_{\Psi_2^1},\ \ C_6 = I_{\Psi_2^2},\ \ C_7 = I_{\beta=1} \ \text{ and } C_8 = I_a. \tag{48b}$$

In (48), besides C_8, the other conserved quantities that can be treated as independent are C_3, C_4, C_5 and C_6. It is easy to show that

$$C_1 = \frac{1}{2}\left(C_3^2 - C_4^2\right) + \frac{1}{2}\left(C_5^2 - C_6^2\right), \tag{49a}$$

$$C_2 = C_3 C_4 + C_5 C_6 \tag{49b}$$

and

$$C_7 = \frac{1}{2}\left(C_3^2 + C_4^2 + C_5^2 + C_6^2\right). \tag{49c}$$

Elimination of p_1 between C_3 and C_4 yields

$$q_1 = \frac{C_4}{\omega}\cos\omega\text{t} - \frac{C_3}{\omega}\sin\omega\text{t}. \tag{50a}$$

Similarly, we have

$$q_2 = \frac{C_6}{\omega}\cos\omega\text{t} - \frac{C_5}{\omega}\sin\omega\text{t}. \tag{50b}$$

Since q_1 and q_2 represents the general solution of the uncoupled Harmonic oscillators in (2a) and (2b), the system is completely specified by the four-parameter Abelian symmetry group generated by G_3, G_4, G_5 and G_6. We have seen that invariance of the system under G_8 leads to the conservation of relative angular momentum in (40c).

4.2. Alternative Representation

In the above we studied the Noether's symmetries of the uncoupled Harmonic oscillator by using the direct Lagrangian given in (3a). We shall now carry out a similar analysis by taking recourse to the use of the alternative Lagrangian in (3b) and examine in some detail how the association between symmetries and conservation laws is affected as we go from direct representation to an inequivalent one.

The Hamiltonian corresponding to the Lagrangian in (3b) is given by (4b). Using this Hamiltonian in (24) we find that $\xi(q_1, q_2, t)$ is not a function of q_1 and q_2 such that it could again be represented by (28). The quantities η_i also formally satisfy the equation in (31) with the exception that (a_{ij}) is now a traceless diagonal matrix. This allows us to write the gauge function f in the form

$$f = \frac{1}{2}q_1 q_2 \ddot{\beta} + \dot{\psi}_1 q_2 + \dot{\psi}_2 q_1. \tag{51}$$

The invariant quantity I can again be represented by (36) but with I_β, I_{ψ_i} and I_a redefined as

$$I_\beta = \frac{1}{2}q_1 q_2 \ddot{\beta} - \frac{1}{2}(q_1 p_1 + q_2 p_2)\dot{\beta} + \left(p_1 p_2 + \omega^2 q_1 q_2\right)\beta, \tag{52a}$$

$$I_{\psi_1} = \dot{\psi}_1 q_2 - \psi_1 p_1, \tag{52b}$$

$$I_{\psi_2} = \dot{\psi}_2 q_1 - \psi_2 p_2 \tag{52c}$$

and

$$I_a = -a_{11}q_1p_1 - a_{22}q_2p_2. \tag{52d}$$

As with our previous analysis each of the I's forms a separate constant with the corresponding J's being given by

$$J_\beta = \frac{1}{2}q_1q_2\left(\dddot{\beta} + 4\omega^2\beta\right), \tag{53a}$$

$$J_{\psi_1} = q_2\left(\ddot{\psi}_1 + \omega^2\psi_1\right), \tag{53b}$$

$$J_{\psi_2} = q_1\left(\ddot{\psi}_2 + \omega^2\psi_2\right) \tag{53c}$$

and

$$J_a = \omega^2\left(a_{11}q_1q_2 + a_{22}q_1q_2\right). \tag{53d}$$

Since (a_{ij}) is traceless ($a_{11} = -a_{22}$), J_a is identically zero. If we choose $a_{11} = 1$, we get

$$I_a = q_2p_2 - q_1p_1. \tag{54}$$

The special values of $\beta(t)$ and $\psi_i(t)$ obtained from (41a) and (41b) are again given by (42a) and (42b). Using these values we can construct eight linearly independent group generators. Except for G_8 all other G_i's, $i = 1, 2, \ldots, 7$, coincide with those given in (47). The generator

$$G_8 = q_1\frac{\partial}{\partial q_1} - q_2\frac{\partial}{\partial q_2} \tag{55}$$

has the corresponding conserved quantity given in (54). Thus $C_8 = I_a$ and we write

$$C_8 = q_2p_2 - q_1p_1. \tag{56}$$

As with (40c) we recognize (56) as the relative angular momentum. Since G_8 is a squeezing operator, the conservation of angular momentum in the present case results from invariance of the system under squeeze. Thus the alternative Lagrangian representation brings in a point of contrast for the association of symmetries and conservation laws with the corresponding result found by using a direct analytic representation. For the direct representation, the angular momentum conservation arises due to invariance of the system under rotation. Although the results for G_i's for $i = 1, 2, \ldots, 7$ in both representations are equal, the corresponding conserved quantities are different in form. The results for C_i's read

$$C_1 = \left(p_1p_2 - \omega^2q_1q_2\right)\cos 2\omega t + \omega\left(q_1p_1 + q_2p_2\right)\sin 2\omega t, \tag{57a}$$

$$C_2 = \left(p_1p_2 - \omega^2q_1q_2\right)\sin 2\omega t - \omega\left(q_1p_1 + q_2p_2\right)\cos 2\omega t, \tag{57b}$$

Table II. Commutation Relations for the Symmetry Generators of the Alternative Lagrangian in (3b). As in Table I Each Element $G_{ij} = [G_i, G_j]$

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
G_1	0	$2\omega G_7$	ωG_4	ωG_3	ωG_6	ωG_5	$2\omega G_2$	0
G_2	$-2\omega G_7$	0	$-\omega G_3$	ωG_4	$-\omega G_5$	ωG_6	$-2\omega G_1$	0
G_3	$-\omega G_4$	ωG_3	0	0	0	0	ωG_4	G_3
G_4	$-\omega G_3$	$-\omega G_4$	0	0	0	0	$-\omega G_3$	G_4
G_5	$-\omega G_6$	ωG_5	0	0	0	0	ωG_6	$-G_5$
G_6	$-\omega G_5$	$-\omega G_6$	0	0	0	0	$-\omega G_5$	$-G_6$
G_7	$-2\omega G_2$	$2\omega G_1$	$-\omega G_4$	ωG_3	$-\omega G_6$	ωG_5	0	0
G_8	0	0	$-G_3$	$-G_4$	G_5	G_6	0	0

$$C_3 = -p_1 \cos\omega t - \omega q_2 \sin\omega t, \tag{57c}$$

$$C_4 = -p_1 \sin\omega t + \omega q_2 \cos\omega t, \tag{57d}$$

$$C_5 = -p_2 \cos\omega t - \omega q_1 \sin\omega t, \tag{57e}$$

$$C_6 = -p_2 \sin\omega t + \omega q_1 \cos\omega t \tag{57f}$$

and

$$C_7 = p_1 p_2 + \omega^2 q_1 q_2. \tag{57g}$$

From (47b), (48b) and (57g) it is clear that the association of invariance under time translation with the conservation of total energy is independent of the choice for Lagrangian representation. We present in Table 2 the algebra of the eight parameter Lie group for the alternative Lagrangian representation.

It is easy to see that in the appropriate limit the results given in Table 2 go over to those for linear one dimensional Harmonic oscillator (Lutzky, 1978). The solution of uncoupled oscillators can be obtained as

$$q_1 = \frac{C_6}{\omega} \cos\omega \mathrm{t} - \frac{C_5}{\omega} \sin\omega \mathrm{t}. \tag{58a}$$

Similarly, we have

$$q_2 = \frac{C_4}{\omega} \cos\omega \mathrm{t} - \frac{C_3}{\omega} \sin\omega \mathrm{t}. \tag{58b}$$

The conserved quantities C_3, C_4, C_5, C_6 are given in (57c)–(57f). Comparison of (50) and (58) exhibits that matrices $\frac{1}{\omega}\begin{pmatrix} C_3 & C_4 \\ C_5 & C_6 \end{pmatrix}$ and $\frac{1}{\omega}\begin{pmatrix} C_5 & C_6 \\ C_3 & C_4 \end{pmatrix}$ operate on the vector $\begin{pmatrix} -\sin\omega t \\ \cos\omega t \end{pmatrix}$ to give the solutions $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$. The interchange of rows in these square matrices merely reflects the difference in the association for Euler-Lagrange equations and equations of the motion for the direct and alternative Lagrangian representations.

5. NOETHER SYMMETRIES OF COUPLED HARMONIC OSCILLATORS

In (6) and (7) we have presented results for direct and alternative analytic representations of the coupled Harmonic oscillators. Studies in the symmetry properties of coupled oscillators have become an active branch of mathematics with application in physics (Han *et al.*, 1995). An oscillator with one type of free vibration has a single natural frequency. On the other hand, two coupled oscillators can exchange energies between them and vibrate in several ways leading to multiple resonant frequencies often called the normal modes of the system. Thus it will be an interesting curiosity to envisage a study on the symmetry analysis of the coupled oscillators with a view to bring out the points of contrast and of similarity between the results of the coupled system with the corresponding results of the uncoupled one. As with our analysis for the uncoupled oscillators we devote two separate subsections to present results for the association between symmetries and conservation laws as obtained by the use of direct and inequivalent Lagrangians.

5.1. Direct Representation

To carry out the symmetry analysis by using the direct representation we use (7a) in (24) to get

$$\begin{aligned}
&\frac{\partial f}{\partial t} + p_1\frac{\partial f}{\partial q_1} + p_2\frac{\partial f}{\partial q_2} \\
&\quad + \frac{1}{2}\left(\frac{\partial \xi}{\partial t} + p_1\frac{\partial \xi}{\partial q_1} + p_2\frac{\partial \xi}{\partial q_2}\right) \\
&\quad \times \left(p_1^2 + p_2^2 + \omega^2 q_1^2 + \omega^2 q_2^2 - 2\alpha q_1 q_2\right) \\
&\quad - \left(\frac{\partial \eta_1}{\partial t} + p_1\frac{\partial \eta_1}{\partial q_1} + p_2\frac{\partial \eta_1}{\partial q_2}\right) p_1 + \eta_1\left(\omega^2 q_1 - \alpha q_2\right) \\
&\quad - \left(\frac{\partial \eta_2}{\partial t} + p_1\frac{\partial \eta_2}{\partial q_1} + p_2\frac{\partial \eta_2}{\partial q_2}\right) p_2 + \eta_2\left(\omega^2 q_2 - \alpha q_1\right) = 0. \qquad (59)
\end{aligned}$$

In the limit of no coupling ($\alpha = 0$), the equation in (59) goes over to that in (25). From (59) we can construct equations similar to those in (27). The coefficients of p_i^0, p_1 and p_2 give

$$p_{1,2}^0 : \quad \frac{\partial f}{\partial t} + \frac{\omega^2}{2}\left(q_1^2 + q_2^2\right)\frac{\partial \xi}{\partial t} - 2\alpha q_1 q_2 \frac{\partial \xi}{\partial t} + \omega^2 \eta_1 q_1 + \omega^2 \eta_2 q_2 - \alpha \eta_1 q_2 - \alpha \eta_2 q_1 = 0, \tag{60a}$$

$$p_1 : \quad \frac{\partial f}{\partial q_1} + \frac{\omega^2}{2}\left(q_1^2 + q_2^2\right)\frac{\partial \xi}{\partial q_1} - \alpha q_1 q_2 \frac{\partial \xi}{\partial q_1} - \frac{\partial \eta_1}{\partial t} = 0 \tag{60b}$$

and

$$p_2 : \quad \frac{\partial f}{\partial q_2} + \frac{\omega^2}{2}\left(q_1^2 + q_2^2\right)\frac{\partial \xi}{\partial q_2} - \alpha q_1 q_2 \frac{\partial \xi}{\partial q_2} - \frac{\partial \eta_2}{\partial t} = 0. \tag{60c}$$

The other equations obtained from the coefficients of $p_1^2, \ldots, p_2^3$ are exactly the same as those given in (27d)–(27j) and lead to (28) and (31). From (28) and (60) we get

$$\frac{\partial f}{\partial t} + \frac{\omega^2}{2}\left(q_1^2 + q_2^2\right)\dot{\beta} - 2\alpha q_1 q_2 \dot{\beta} + \eta_1\left(\omega^2 q_1 - \alpha q_2\right) + \eta_2\left(\omega^2 q_2 - \alpha q_1\right) = 0, \tag{61}$$

and two other equations which are exactly the same as those given in (33) and (34) such that the value of f is once again given by (35).

We now use the values of H and f form (7a) and (35) in (22) to get the invariant in the form (36). In this case

$$I_\beta = \frac{1}{4}\left(q_1^2 + q_2^2\right)\ddot{\beta} - \frac{1}{2}\left(q_1 p_1 + q_2 p_2\right)\dot{\beta} + \frac{1}{2}\left(p_1^2 + p_2^2 + \omega^2 q_1^2 + \omega^2 q_2^2 - 2\alpha q_1 q_2\right)\beta. \tag{62}$$

The results for I_{ψ_i} and I_a coincide with the the expressions in (37b) and (37c). In close analogy with our treatment of the uncoupled Harmonic oscillators, it is straightforward to see from (35) and (61) that each of I_β, I_{ψ_1}, I_{ψ_2} and I_a forms a separate constant. It is interesting to note that the effect of coupling on the conserved quantity enters only through I_β. From (41a) and (41b)

$$\beta = 1, \quad \beta = e^{\pm 2\omega_1 t}, \quad \beta = e^{\pm 2\omega_2 t} \tag{63a}$$

and

$$\psi_i^\pm = e^{\pm \omega_1 t}, \quad \psi_i^\pm = e^{\pm \omega_2 t} \tag{63b}$$

where $\omega_1 = \sqrt{\omega^2 - \alpha}$ and $\omega_2 = \sqrt{\omega^2 + \alpha}$ stand for the resonant frequencies of the coupled oscillators. In the absence of interaction the values of β and ψ_i in (63) go over to those in (42). For the values of β and ψ_i given in (63a) and (63b) we have obtained following conserved quantities

$$C_1 = \mathrm{Re} I_{\beta = e^{+2i\omega_1 t}}$$

$$= \frac{1}{2}\left(p_1^2 + p_2^2 - \omega^2 q_1^2 - \omega^2 q_2^2 - 2\alpha\left(q_1 q_2 - q_1^2 - q_2^2\right)\right) \cos 2\omega_1 t$$

$$+\omega_1 \left(q_1 p_1 + q_2 p_2\right) \sin 2\omega_1 t, \tag{64a}$$

$$C_2 = \mathrm{Im} I_{\beta = e^{+2i\omega_1 t}}$$
$$= \frac{1}{2}\left(p_1^2 + p_2^2 - \omega^2 q_1^2 - \omega^2 q_2^2 - 2\alpha\left(q_1 q_2 - q_1^2 - q_2^2\right)\right) \sin 2\omega_1 t$$

$$-\omega_1 \left(q_1 p_1 + q_2 p_2\right) \cos 2\omega_1 t, \tag{64b}$$

$$C_3 = \mathrm{Re} I_{\psi_1 = e^{+i\omega_1 t}} = -p_1 \cos\omega_1 t - \omega_1 q_1 \sin\omega_1 t, \tag{64c}$$

$$C_4 = \mathrm{Im} I_{\psi_1 = e^{+i\omega_1 t}} = -p_1 \sin\omega_1 t + \omega_1 q_1 \cos\omega_1 t, \tag{64d}$$

$$C_5 = \mathrm{Re} I_{\psi_2 = e^{+i\omega_1 t}} = -p_2 \cos\omega_1 t - \omega_1 q_2 \sin\omega_1 t, \tag{64e}$$

$$C_6 = \mathrm{Im} I_{\psi_2 = e^{+i\omega_1 t}} = -p_2 \sin\omega_1 t + \omega_1 q_2 \cos\omega_1 t. \tag{64f}$$

$$C_7 = I_{\beta=1} = \frac{1}{2}\left(p_1^2 + p_2^2 + \omega^2 q_1^2 + \omega^2 q_2^2 - 2\alpha q_1 q_2\right), \tag{64g}$$

$$C_8 = I_a = q_1 p_2 - q_2 p_1, \tag{64h}$$

$$C_9 = \mathrm{Re} I_{\beta = e^{+2i\omega_2 t}}$$

$$= \frac{1}{2}\left(p_1^2 + p_2^2 - \omega^2 q_1^2 - \omega^2 q_2^2 - 2\alpha(q_1 q_2 + q_1^2 + q_2^2)\right) \cos 2\omega_2 t$$

$$+\omega_2 \left(q_1 p_1 + q_2 p_2\right) \sin 2\omega_2 t, \tag{64i}$$

$$C_{10} = \mathrm{Im} I_{\beta = e^{+2i\omega_2 t}}$$

$$= \frac{1}{2}\left(p_1^2 + p_2^2 - \omega^2 q_1^2 - \omega^2 q_2^2 - 2\alpha\left(q_1 q_2 + q_1^2 + q_2^2\right)\right) \sin 2\omega_2 t$$

$$-\omega_2 (q_1 p_1 + q_2 p_2) \cos 2\omega_2 t, \tag{64j}$$

$$C_{11} = \mathrm{Re} I_{\psi_1 = e^{+i\omega_2 t}} = -p_1 \cos\omega_2 t - \omega_2 q_1 \sin\omega_2 t, \tag{64k}$$

$$C_{12} = \mathrm{Im} I_{\psi_1 = e^{+i\omega_2 t}} = -p_1 \sin\omega_2 t + \omega_2 q_1 \cos\omega_2 t, \tag{64l}$$

$$C_{13} = \mathrm{Re} I_{\psi_2 = e^{+i\omega_2 t}} = -p_2 \cos\omega_2 t - \omega_2 q_2 \sin\omega_2 t \tag{64m}$$

and

$$C_{14} = \mathrm{Im} I_{\psi_2 = e^{+i\omega_2 t}} = -p_2 \sin\omega_2 t + \omega_2 q_2 \cos\omega_2 t. \tag{64n}$$

Fourteen linearly independent symmetry generators associated with the above conserved quantities are given by

$$G_1 = \cos 2\omega_1 t \, \frac{\partial}{\partial t} - \omega_1 q_1 \sin 2\omega_1 t \, \frac{\partial}{\partial q_1} - \omega_! q_2 \sin 2\omega_1 t \, \frac{\partial}{\partial q_2}, \tag{65a}$$

$$G_2 = \sin 2\omega_1 t \, \frac{\partial}{\partial t} + \omega_1 q_1 \cos 2\omega_1 t \, \frac{\partial}{\partial q_1} + \omega_1 q_2 \cos 2\omega_1 t \, \frac{\partial}{\partial q_2}, \tag{65b}$$

$$G_3 = \cos\omega_1 t \frac{\partial}{\partial q_1}, \tag{65c}$$

$$G_4 = \sin\omega_1 t \frac{\partial}{\partial q_1}, \tag{65d}$$

$$G_5 = \cos\omega_1 t \frac{\partial}{\partial q_2}, \tag{65e}$$

$$G_6 = \sin\omega_1 t \frac{\partial}{\partial q_2}, \tag{65f}$$

$$G_7 = \frac{\partial}{\partial t}, \tag{65g}$$

$$G_8 = q_2 \frac{\partial}{\partial q_1} - q_1 \frac{\partial}{\partial q_2}, \tag{65h}$$

$$G_9 = \cos 2\omega_2 t \, \frac{\partial}{\partial t} - \omega_2 q_1 \sin 2\omega_2 t \, \frac{\partial}{\partial q_1} - \omega_2 q_2 \sin 2\omega_2 t \, \frac{\partial}{\partial q_2}, \tag{65i}$$

$$G_{10} = \sin 2\omega_2 t \, \frac{\partial}{\partial t} + \omega_2 q_1 \cos 2\omega_2 t \, \frac{\partial}{\partial q_1} + \omega_2 q_2 \cos 2\omega_2 t \, \frac{\partial}{\partial q_2}, \tag{65j}$$

$$G_{11} = \cos\omega_2 t \frac{\partial}{\partial q_1}, \tag{65k}$$

$$G_{12} = \sin\omega_2 t \frac{\partial}{\partial q_1}, \tag{65l}$$

$$G_{13} = \cos\omega_2 t \frac{\partial}{\partial q_2} \tag{65m}$$

and

$$G_{14} = \sin\omega_2 t \frac{\partial}{\partial q_2}. \tag{65n}$$

As expected, in the limit of no coupling, we get from (64a)–(64n) only eight linearly independent generators as given in (47). In this context, we also note that the conserved quantity C_7 represents the total energy of the coupled oscillators and is associated with the time translation invariance of the system. The invariant quantity C_8 stands for the relative angular momentum. Since G_8 represents a rotation operator, here conservation of angular momentum arises due to rotational invariance. The commutation relations for the generators of the coupled oscillators are schematically shown in the Table III.

In this Table $\mathbf{A}(\omega_1)$ represents the entries in Table I with the last two rows and columns being deleted. Also we have to use ω_1 for ω. The square array of commutators in $\mathbf{A}(\omega_2)$ carry a similar meaning. When the 2×2 null array is appropriately included, the rectangular arrays $\mathbf{B}(\omega_1)$ and $\mathbf{B}(\omega_2)$ will stand for the rows and columns which were deleted in writing $\mathbf{A}(\omega_1)$ and $\mathbf{A}(\omega_2)$. In contrast to the commutators in $\mathbf{A}$'s and $\mathbf{B}$'s, the commutators in $\mathbf{C}$ depend on both ω_1 and ω_2 and we have used $\mathbf{C} = \mathbf{C}(\omega_\mathbf{1}, \omega_\mathbf{2})$. Amongst the square array of commutators contained in $\mathbf{C}(\omega_1, \omega_2)$ we have

$$[G_i, G_j] = 0 \quad \text{for} \;\; i = 3, \ldots, 6, \quad j = 11, \ldots, 14. \tag{66}$$

The other commutators, $[G_i, G_j]$, $i = 1, 2$, $j = 9, \ldots, 14$ go over to the entries in Table I for $[G_i, G_j]$, $i = 1, 2$, $j = 1, \ldots, 6$. Similarly $[G_i, G_j]$, $i = 3, \ldots, 6$, $j = 9, 10$ coincide with the commutators $[G_i, G_j]$, $i = 3, \ldots, 6$, $j = 1, 2$ in Table I. This serves as a useful check on the symmetry analysis presented for the coupled oscillators using the Hamiltonian of the direct representation.

Table III. Commutation Relations for Generators in (65)

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}	G_{11}	G_{12}	G_{13}	G_{14}
G_1														
G_2														
G_3			$\mathbf{A}(\omega_1)$					$\mathbf{B}(\omega_1)$			$\mathbf{C}(\omega_1, \omega_2)$			
G_4														
G_5														
G_6														
G_7			$-\mathbf{B}(\omega_1)$				0	0			$-\mathbf{B}(\omega_2)$			
G_8							0	0						
G_9														
G_{10}														
G_{11}			$-\mathbf{C}(\omega_1, \omega_2)$					$\mathbf{B}(\omega_2)$			$\mathbf{A}(\omega_2)$			
G_{12}														
G_{13}														
G_{14}														

5.2. Alternative Representation

Here we work with the Hamiltonian in (7b). For this Hamiltonian f, I_{ψ_1}, I_{ψ_2} and I_a come out exactly in the same form as given in (51), (52b), (52c) and (52d) for the alternative representation of the uncoupled Harmonic oscillators. The effect of the coupling enters only through I_β written as

$$I_\beta = \frac{1}{2}q_1q_2\ddot{\beta} - \frac{1}{2}(q_1p_1 + q_2p_2)\dot{\beta}$$

$$+ \left(p_1p_2 + \omega^2 q_1q_2 - \frac{\alpha}{2}(q_1^2 + q_2^2)\right)\beta, \tag{67}$$

with values of β given in (63a). For these β values we find the conserved quantities

$$C_1 = \mathrm{Re} I_{\beta=e^{+2i\omega_1 t}}$$

$$= \left(p_1p_2 - \omega^2 q_1q_2 - \frac{\alpha}{2}\left(q_1^2 + q_2^2 - 4q_1q_2\right)\right) \cos 2\omega_1 t$$

$$+ \omega_1 (q_1p_1 + q_2p_2) \sin 2\omega_1 t, \tag{68a}$$

$$C_2 = \mathrm{Im} I_{\beta=e^{+2i\omega_1 t}}$$

$$= \left(p_1p_2 - \omega^2 q_1q_2 - \frac{\alpha}{2}\left(q_1^2 + q_2^2 - 4q_1q_2\right)\right) \sin 2\omega_1 t$$

$$- \omega_1 (q_1p_1 + q_2p_2) \cos 2\omega_1 t, \tag{68b}$$

$$C_7 = I_{\beta=1} = p_1p_2 + \omega^2 q_1q_2 - \frac{\alpha}{2}(q_1^2 + q_2^2), \tag{68c}$$

$$C_9 = \mathrm{Re} I_{\beta=e^{+2i\omega_2 t}}$$

$$= \left(p_1p_2 - \omega^2 q_1q_2 - \frac{\alpha}{2}\left(q_1^2 + q_2^2 + 4q_1q_2\right)\right) \cos 2\omega_2 t$$

$$+ \omega_2 (q_1p_1 + q_2p_2) \sin 2\omega_2 t \tag{68d}$$

and

$$C_{10} = \mathrm{Im} I_{\beta=e^{+2i\omega_2 t}}$$

$$= \left(p_1p_2 - \omega^2 q_1q_2 - \frac{\alpha}{2}\left(q_1^2 + q_2^2 + 4q_1q_2\right)\right) \sin 2\omega_2 t$$

$$- \omega_2 (q_1p_1 + q_2p_2) \cos 2\omega_2 t, \tag{68e}$$

with the corresponding generators given in (65a), (65b), (65g), (65i)and (65j), respectively. In particular, G_7 in (65g) is a time translation operator such that C_7 represents the total energy of the system. Comparison of the results in (64g) and (68c) clearly shows that the total energy of the system is expressed in different ways for the direct and alternative phase-space representations. We have noted above that (52d) remains invariant as we go from uncoupled to coupled representations. This implies that we have again $C_8 = q_2 p_2 - q_1 p_1$ as the relative angular momentum of the coupled system. The corresponding generator is the squeezing operator given in (55).

6. SUMMARY AND CONCLUDING REMARKS

Noether's theorem provides a one-to-one correspondence between the symmetry properties and conserved quantities of a dynamical system. We have chosen to work with a theoretical framework which attributes the reason for this to the properties of some auxiliary equations which can always be written in the form of a total time derivative. This realization for the auxiliary equations allowed us to examine how the association between symmetries and conservation laws changes as one alters the Lagrangian representation of the system.

We carried out the Noether or variational symmetry analysis for two uncoupled Harmonic oscillators as well as two such oscillators coupled by an interaction. Each of these newtonian systems can be analytically represented by two different Lagrangians which are not connected by a gauge term. These are the so-called alternative Lagrangians. For brevity, we called one of the Lagrangians as direct and the other as alternative. Irrespective of whether the representation is direct or alternative, we found that there are eight conserved quantities for the uncoupled harmonic oscillators. We also worked out the generators of the symmetry transformations that lead to these conserved quantities and studied the associated Lie algebra. This allowed us to construct the solution of the system. For the direct representation we found that the conservation of total energy follows from invariance of the system under time translation. This is also true for the alternative Lagrangian representation. But so far as the conservation of angular momentum is concerned the situation is quite different. For the direct representation the conservation of angular momentum follows from rotational invariance while for the alternative representation such a conservation is related with invariance of the system under squeeze.

As opposed to the uncoupled system, the coupled oscillators are characterized by fourteen invariant quantities. Referring to the conserved quantities in (64), we see that, in the limit of no coupling, C_1, C_2, C_3, C_4, C_5 and C_6 coincide with C_9, C_{10}, C_{11}, C_{12}, C_{13} and C_{14} respectively and finally, we are left with only eight conserved quantities as given in (48). This comment made in the context of direct analytic representation is equally true for the alternative representation. It is found that the interaction does not affect the association between symmetries and conservation laws.

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